

# A RULER AND COMPASS CONSTRUCTION OF THE 360 DEGREES SPHERICAL PERSPECTIVE

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**ABSTRACT.** We describe a general setup for anamorphosis and perspective, then obtain a ruler and compass construction for the 360 degrees spherical perspective. We consider its uses in freehand drawing and computer visualization, and its relation with reflections on a sphere.

## 1. INTRODUCTION

It is often useful in the visual arts to depict a scene composed within a very wide angle of view. For scenes that are wide but not tall (lanscapes) one can project on a cylinder and then unfold this developable surface isometrically onto a plane to get the so-called cylindrical or panoramic perspective. When the scene is wide all around an axis, it can be projected onto a half-sphere and then deformed onto a disc on the plane. This is the so-called spherical perspective, and has been described thoroughly in [1] in the 1960s. It is in fact a hemi-spherical perspective, that allows a depiction of up to 180 degrees around an axis, wherein images of lines can be easily constructed with ruler and compass within a reasonable approximation. It is often useful, however, to depict an even wider scene. The author, having taught a course in art and mathematics for a few years to a varied audience (urban sketchers, architects, school teachers, programmers) has often received requests from students regarding two questions: how to plot (either freehand or with minimal instruments) a view wider than the 180° allowed by (hemi)spherical perspective, and how to draw a sphere reflection. This paper solves the former question and relates it to the latter. It also hopes to help clarify some concepts of anamorphosis and perspective.

## 2. PREVIOUS WORKS ON WIDE ANGLE PERSPECTIVES

In their 1968 work, Barre and Flocon [1] described a ruler and compass method to plot a 180 degrees spherical perspective. This was a work with a focus on the artistic practice of actual freehand drawing. Since then several works of a computational nature have proposed various types of wide angle perspectives, by expanding the angle of view up to 360 degrees or by generalizing the shapes of the projection surfaces ([4] and [5]). These works are of a computational nature and not concerned with the artistic practice of freehand or ruler and compass drawing. As far as this author can tell, there has been no publication wherein a system is proposed to allow a depiction of a 360 degree spherical perspective in a way adequate for drawing from observation or from orthogonal plans with the use of minimal equipment such as ruler and compass.

Regarding contributions by artists themselves, Dick Thermes is well-known for his paintings on spherical surfaces. He has published a book [6] on the subject. His approach is based on gridding in the manner that follows from [1]. His solution to go

beyond the 180 degrees is simply to draw the two complementary 180 degree views separately and place them adjacently to each other. When drawing on a sphere he can put each view on on its hemisphere, but that is a work of anamorphosis and not of perspective proper.

The most common artistic device for representating views beyond 180 degrees is that of drawing sphere reflections from observation, and we will consider the relation of these with spherical perspective.

In [2] sphere reflections have been proposed as way to obtain a wide field of view (with difficulties that we will discuss ahead). Again this is a work of a computational nature.

### 3. PERSPECTIVES

Perspectives are ways of representing spatial scenes on a plane, with relation to an observer.

We take a *scene* to mean any subset of the euclidean 3-dimensional space. We represent an *observer* by a point in 3-space, usually denoted by  $O$ .

Our purpose is to map each point of a scene in 3-space onto a point on a plane, the latter point being called the perspective image of the former.

Mostly we will be concerned with sets of points, lines, planes, or circles, and will use these to approximate more complicated objects. A map from  $\mathbb{R}^3$  to  $\mathbb{R}^2$  that fails to project these elementary objects with simplicity is not adequate for the purposes of a perspective to be used "by hand".

All the maps that are usually called perspectives (plane, cylindrical, spherical) can be constructed as particular cases of a same scheme that we will now describe.

**Definition 3.1.** Let  $O$  be an observer and  $P$  a point in a scene. We call the ray  $\overrightarrow{OP}$  the *ray of sight* from  $O$  to  $P$ . Let  $S_O^2$  be the unit sphere centered at  $O$ . We call  $\overrightarrow{OP} \cap S^2$  the *direction of sight* from  $O$  to  $P$ .<sup>1</sup>

A point  $P$  in a scene will define a vector  $\overrightarrow{OP}$  and a ray with the same notation. We will let the ambiguity stand and let context distinguish them. The unit vector  $\overrightarrow{OP}/||\overrightarrow{OP}||$  on the sphere can be seen as the equivalence class of all the points that have it's direction of sight from  $O$ , i.e., all the points in the ray of sight  $\overrightarrow{OP}$ .

. Now recall a few generalities about circles on spheres:

A *great circle* is a circle on a sphere, defined by the intersection of the sphere with a plane through the origin.

Given a point  $P$  on the sphere we call *antipode point* of  $P$  to the its diametrically opposite point on the sphere.

Each point  $P$  on the sphere defines a family of great circles that covers the sphere, all of those circles crossing both  $P$  and its antipode.

Each two non-antipodal points  $P$  and  $Q$  on the sphere define a unique great circle, the intersection of the sphere with the plane  $POQ$ .

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<sup>1</sup>The general point is that for an observer at  $O$  points will be equivalent if they are radial from  $O$ . These equivalence classes are naturally formalized into points of the projective space in classical perspective. For more general perspectives, however we have to take into account the need to represent both a direction and its diametrically opposite, hence the sphere is the natural manifold of directions.

A *meridian* is one contiguous half of a great circle. Given a point  $P$  and its antipode  $\bar{P}$  we will call  $P$ -meridian or  $P\bar{P}$ -meridian to a meridian that is an arc of a great circle whose endpoints are at  $P$  and at its antipode.

We are now ready to consider the problem of vanishing points.

**Example 3.2.** Let  $l$  be a line not crossing  $O$ . There is a single plane  $H$  through  $O$  containing  $l$ . This plane projects radially into a great circle  $C$  on the sphere. The set of rays of sight of  $l$  will form a cone, the set  $C_O(l) = \{\overrightarrow{OP} : P \in l\}$  which is a half-plane contained in  $H$ . The boundary of this half-plane is the line  $l_O$ , the translation of  $l$  to the origin. This line corresponds to two rays from  $O$  none of which is a ray of sight of an actual point of  $l$  but correspond to the limit of the directions of sight of an observer that follows the line to infinity in both directions. The intersection of  $C_O(l)$  with the sphere is the set of directions of sight of  $l$ . It is half of the great circle  $C$ , and it does not contain the two antipodal endpoints, which are the intersection of  $l_O$  with the sphere.

The two missing directions in example 3.2 correspond to the intuitive notion of vanishing point. As the eye follows a line  $r$  to infinity, the ray of sight will in the limit become parallel to the line it follows. This will happen at both ends of the line  $r$ . But since this happens for no actual point on the line, the projection of the line  $r$  on the sphere of directions will be missing two points at its end. Such missing points, for lines or more complex sets, can be added to the cone of directions by taking the topological closure.

**Definition 3.3.** Let  $\mathcal{S}$  be a scene. Let  $O$  be an observer. Let  $\bar{\mathcal{S}}$  be the topological closure of  $\mathcal{S}$ . The rays of sight of the points of  $\bar{\mathcal{S}}$  define a cone  $C_O(\bar{\mathcal{S}})$  with vertex at  $O$ . Let  $\mathcal{C}_O(\mathcal{S}) = \overline{C_O(\bar{\mathcal{S}})} \cap S_O^2$  be the closure of the intersection of this cone with the unit sphere  $S_O^2$ . We call  $\mathcal{C}_O(\mathcal{S})$  the *cone of sight* of  $\mathcal{S}$  relative to  $O$ . We will abuse the term to also mean the corresponding cone of rays in 3-space stemming from  $O$  that project radially to  $\mathcal{C}_O(\mathcal{S})$ , and we will denote that cone of rays by  $\tilde{\mathcal{C}}_O(X)$ .

Taking the topological closure of the cone of sight allows us to obtain its vanishing points in the following definition. Note that we took the closure of  $\mathcal{S}$  in order to avoid the appearance of "false" vanishing points. In what follows we will always assume the scenes to be closed sets, so this step may be ignored.

**Definition 3.4.** We call *vanishing points* of a scene  $\mathcal{S}$  to the set  $\mathcal{C}_O(\mathcal{S}) \setminus C_O(\bar{\mathcal{S}})$ .

That is, vanishing points are the frontier points of the cone of directions that are not directions of actual points in (the closure of) the scene.

**Example 3.5.** Going back to example 3.2 we take the closure of the half circle and obtain its two endpoints. We see that the set of vanishing points of a line  $l$  is the intersection of its translation to the origin,  $l_O$ , with the sphere of directions. The cone of sight of  $l$  is one half of the great circle of  $H$ , including its end points. The corresponding cone of rays is the half-plane of  $H$  plus the set of two diametrically opposite rays passing through the vanishing points.

**Example 3.6.** Let  $H$  be a plane not passing through  $O$ . The cone of rays of the individual points of  $H$  forms a half-space whose frontier is a plane parallel to  $H$  passing through  $O$ . This frontier is not contained in the set of rays of sight of individual points of  $H$ . It is however contained in the closure of that set. On the

sphere, the cone of sight will be a hemisphere and the set of vanishing points will be the great circle that forms its boundary. (In classical perspective the visible part of this circle will form a line, called a vanishing line.)

**Definition 3.7.** We say that a surface  $S$  is radial with respect to a point  $O$  if any ray stemming from  $O$  will hit the surface on at most one point. We will say that a surface is radial if the point  $O$  is obvious from context.

**Definition 3.8.** A *perspective*<sup>2</sup> is a map from euclidean 3-space to a region of the plane, that is achieved by the composition of two maps. The first we call an *anamorphism* and the second a *flattening*.

Given an observer  $O$ , a surface  $S$  that is radial for  $O$ , and a scene  $X$ , we say that  $\tilde{C}_O(X) \cap S$  is the *anamorphosis* (or *anamorphic image*) of  $X$  on  $S$  relative to  $O$ . To the map that takes each point to its anamorphic image we call the anamorphism onto  $S$  relative to  $O$ .<sup>34</sup>

A *flattening* is a map from  $S$  to a region of the plane  $\mathbb{R}^2$ . It takes the anamorphosis of a scene to its perspective image.

A perspective is a map  $\pi \circ \Lambda$  where  $\Lambda : \mathbb{R}^3 \rightarrow S$  is an anamorphosis and  $\pi : S \rightarrow \mathbb{R}^2$  is a flattening.

We note that the anamorphism is fully defined by the choice of surface  $S$  and observer  $O$ . The map itself is just the conical projection from  $O$  onto that surface.

**Example 3.9.** In classical perspective the anamorphic surface is a plane and the flattening can be seen as the identity map up to scaling. Lines will project onto lines, but for each line only at most one of the vanishing points will be present in the perspective image. Lines that are parallel to the anamorphic plane will have no vanishing points at all in their perspective image.

**Example 3.10.** in cylindrical perspective,  $S$  is a cylinder with  $O$  on the axis, and the flattening consists in unrolling the cylinder over a plane. The anamorphic images of straight lines will be arcs of ellipses and the flattening will unroll them onto sinusoids. The pairs of vanishing points of lines on the sphere of directions will project onto the cylinder except if their directions are along the axis of the cylinder.

**Example 3.11.** In the the so-called spherical perspective of [1]  $S$  is a hemisphere of a sphere around  $O$  and the flattening is a restriction to a hemisphere of the azimuthal equidistant projection. Lines will project onto arcs of great circle on the hemisphere, and will have one vanishing point on both the anamorphic image and the perspective image; or two points if these are on the equator.

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<sup>2</sup>We should perhaps be less ambitious and say a *radial* or a *central* perspective. But in the context of this paper it will do.

<sup>3</sup>In intuitive terms, the anamorphosis of a scene onto a surface relative a point  $O$  is an image painted on  $S$  such that the observer at  $O$  would mistake that image for the spatial scene itself. We should notice here that we are using the word in two senses: there is "anamorphosis" as the problem of *tromp l'oeil*, and there is anamorphosis as the mathematical construction we just defined. That the latter is the solution of the former cannot be demonstrated mathematically; it is an empirical fact that depends on the approximate validity of linear optics and on the physiology vision.

<sup>4</sup>The most important thing to note is that on the very definition of anamorphosis the observation point must be specified. Forgetting this leads to all sorts of misconceptions; for instance, *perspective deformation*, a much abused term, is a misnomer for the consequences of having a viewer look at a plane anamorphosis from the wrong observation point.

**Example 3.12.** In the the full spherical perspective considered ahead  $S$  is a sphere around  $O$  and the flattening is the azimuthal equidistant projection. Lines will project onto halves of great circle on the sphere, and will have always two vanishing points on the anamorphic image. Because the anamorphic surface is a sphere, just like the sphere of directions, the anamorphism is a homothety, and can be identified with the identity map.

. We note an interesting symmetry between spherical and plane perspective. In classical perspective the flattening is trivial but the anamorphosis is not. In spherical perspective the reverse is true. This is because in classical perspective the plane of the anamorphosis can be identified with the plane of the perspective, while in the spherical perspective the anamorphic sphere can be identified with the sphere of directions, so the flattening in the former case and the anamorphosis in the latter can be identified with the identity map.

. We have not placed any constraints on the nature of  $S$  or on the maps. The fact is that in artistic practice there are examples where the surface  $S$  is anything from a set of disconnected planes to a myriad of dust-like particles in suspension.  $S$  is certainly not a necessarily a smooth, nor a connected surface, although it is so in all the usual perspectives such as the one we treat here.

#### 4. SPHERICAL PERSPECTIVE

We will now define our spherical perspective, within the general scheme outlined above.

First we must define the anamorphic surface and the place of our observer. We will take  $S$  to be a spherical surface of arbitrary positive radius  $R$ , with the observer  $O$  at its center.

Choosing these elements fully defines our anamorphism. What can we say about it?

As we already discussed, the anamorphic sphere being a surface, the anamorphic image coincides with the cone of sight in the sphere of directions.

Hence the anamorphosis of a line  $l$  onto a sphere relative to its center is an arc of a great circle: the rays of sight from  $O$  to the points of the line  $l$  define (half of) a plane that crosses  $O$ . The anamorphic image of that plane is a great circle, and the image of the line is a half of the great circle, delimited by two vanishing points, determined thus:

If  $r$  is a line, and  $r_O$  is the translation of line  $r$  that passes through  $O$ , then the vanishing points of  $r$  are the intersection of  $r_O$  with  $S$ .

**4.1. The Flattening.** We will now define our flattening map.

We must first define a system of coordinates. We consider a ray stemming from  $O$ , representing a privileged direction of sight. We call it the *central ray of sight* and to its axis we call the *central axis of sight*. We place an orthonormal reference frame  $xyz$  in  $O$ , such that the positive side of the  $y$  axis coincides with the central ray of sight. For quick reference we name the points where the three axes cut the sphere: we call **North** to the intersection of the central ray of sight with the sphere and **South** to its antipode point; **East** to the point where the  $x$  axis touches the sphere and **West** to its antipode; **Zenith** to the point where the positive  $z$  axis touches the sphere and **NaDir** to its antipode, and we represent these points by the letters that we wrote in bold.

We call the plane orthogonal to the central axis of sight (the  $xz$ -plane) the *observer's plane* ; we call the  $yz$ -plane the *sagital plane* . The observer's plane intersects the sphere in a great circle we call the *equator* .

We call the half-space  $y > 0$  the anterior half-space (representing everything in front of the observer) and to the half-space  $y < 0$  we call the posterior half-space (representing all that is behind the observer).

We construct a flattening map<sup>5</sup> that will project all the points of the sphere onto a disc on a plane, with the exception of the south pole. This map composed with the anamorphosis will result in a perspective that images all points of 3-space except those points located on the  $\overrightarrow{OS}$  ray.

We define the flattening thus:

We map the sphere minus its south pole one-to-one to a disc on the plane in such a way that each North-South meridian goes to a line segment and

- i) Distances are preserved along each meridian.
- ii) Angles between meridians are preserved at point  $N$ .

Condition i) means that the map is an isometry for each meridian separately. Since distances measured along great circles of the sphere are proportional to angles from the center, this means that if  $P, Q$  are points on the same meridian and if  $P', Q'$  are their images, then  $|P'Q'| = \angle(POQ)$  up to multiplication by a scale factor<sup>6</sup>. Condition i) also implies that  $N$  will be mapped to the center of the disc with the segments corresponding to the meridians radiating from it.

The lines stemming from the image of  $N$  we will call measuring lines, because angular distances are preserved along them. They will be the essential tool for our plotting of points.

Condition ii) means that the angles between measuring lines at the image of  $N$  will be equal to the angles of their meridians at  $N$  (which are the angles between the planes that define them). It ensures the meridians will be distributed radially preserving their tangents on  $N$ , that is, they will look as if orthogonally dropped onto the tangent plane to the sphere at  $N$ . We call *longitude* of an N-meridian to the angle between its tangent and that of the East meridian at point  $N$ . The longitude of a meridian will equal the angle of its measuring line with the *NE* measuring line.

Condition i) and ii) together imply that the two meridians of each great circle through  $N$  form a diameter of the perspective disc and that distances are preserved along interior points of these diameters.

In intuitive terms, we look at the North-South meridians as inextensible threads, connected to the sphere on the North and South poles. We cut them free at the South pole only, and, pulling them straight along their tangents at  $N$ , make from them a disc on the plane tangent to the sphere at  $N$ . At the end, the threads are radiating from  $N$ , each under the orthogonal shadow of its former position, so that the angles their tangents make at  $N$  are the same as before. Distances have not changed between points on the same thread. The threads together form a disc of radius  $\pi R$ , where  $R$  is the radius of the sphere, with  $N$  at the center and each

<sup>5</sup>This is a construction for the *azimuthal equidistant projection* , well known to cartographers and astronomers, better known in France as the *Postel* projection. A restriction of this map to a hemisphere is used in [1].

<sup>6</sup>For points on the images of these meridians we will freely abuse notation and write equalities between angles and linear measures such as  $|\overline{XZ}| = |\overline{XY}| + 180^\circ$  to mean that these equalities are valid modulo product by the adequate scale factors.

point of the sphere corresponds to a single point on the disc, except for the South pole, that was blown-up and is missing from the outer edge of each thread, hence making the disc open at its boundary. We take the closure of the disc and get a frontier circle which we call the South circle, or the blow-up of the South pole, each of whose points corresponds to one of the directions that the original pole could be approached from (each of them can be identified with one of the meridians, or with a ray of the tangent plane at  $S$ ).

When context is clear we will use the same letters for the  $N, S, E, W, Z, D$  points and their images, and will call equator to the perspective image of the equator. See fig. 1 for a picture of the perspective image of these points and lines.

Points in the anterior half-space will be projected by the anamorphosis into the hemisphere in front of the equator, and these will be flattened in turn onto the inner part of the perspective disc that is contained within the image of the equator. This inner disc will have half of the radius of the perspective disc. The posterior half space will be projected onto the outer ring between the image of the equator and the circle of the blow-up of the south pole  $S$ .

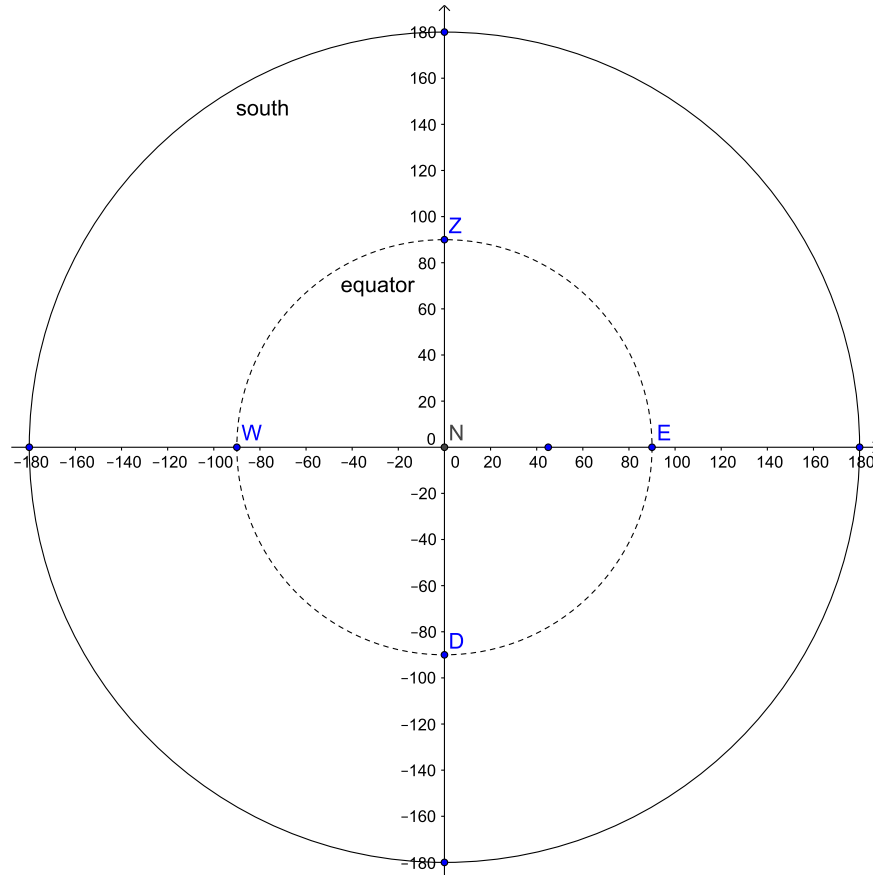


FIGURE 1. The perspective disc with angles marked along the horizontal and vertical measuring lines.

In terms of the  $(x, y, z)$  coordinates, the flattening composed with anamorphosis  $r \mapsto r/||r||$  gives the perspective map  $f : S \mapsto D$  where  $S$  is the sphere of radius  $R$  and  $D$  is a disc of radius  $\pi R$ ,

$$(4.1) \quad f(x, y, z) = \frac{(x, z)}{\sqrt{x^2 + z^2}} \arccos \left( \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right)$$

that is, we project orthogonally against the  $xz$ -plane, take the unit vector, and then scale to a length equal to the value of the angle  $\angle PON$ .

The natural set of spherical coordinates for this map is  $(\rho, \lambda, \theta)$  where

$$(4.2) \quad \begin{aligned} \rho &= |OP| \\ \lambda &= \angle PON = \arccos \left( \frac{y}{|OP|} \right) \\ \theta &= \arccos \left( \frac{x}{\sqrt{x^2 + z^2}} \right) \end{aligned}$$

that is,  $\lambda$  is the latitude and  $\theta$  is the longitude with respect to the north pole. In these coordinates

$$f(x, y, z) = \lambda(\cos(\theta), \sin(\theta))$$

We see that the perspective map does not in fact depend on the norm  $\rho$  of the point, which is to be expected since the anamorphosis is a central projection.

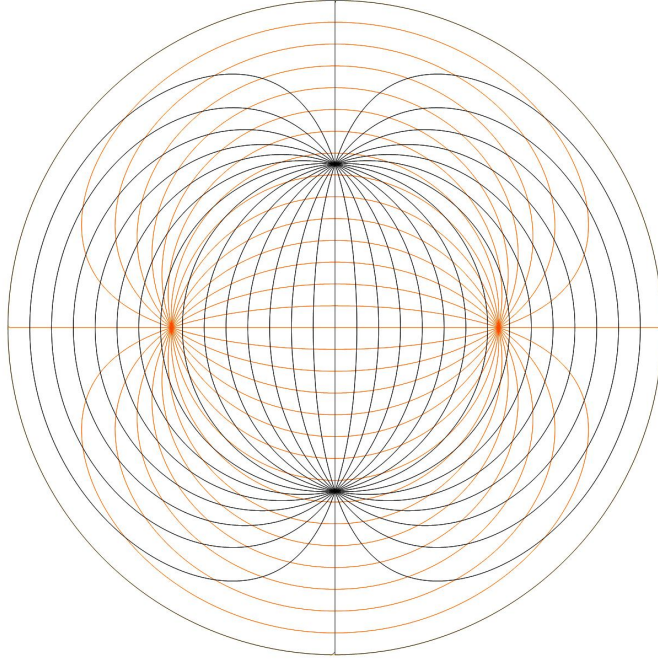


FIGURE 2. 360 degree perspective of vertical and horizontal lines from equation 4.1



## 5. SOLVING A SCENE WITH RULER AND COMPASS

*Solving a scene* in perspective means finding the perspective images of all points on the scene - classically, and in the interest of solving a scene with simple instruments, we are concerned with the images of points grouped into lines, and especially with their vanishing points. We will now show how to solve a scene in our 360 degree spherical perspective using ruler and compass.

A common technique to solve scenes in classical perspective is to make the plane of the perspective image do double or triple duty by superposing on various orthogonal projections. This technique also works in spherical perspective. We will illustrate it in the following construction that we will make use of in the next section.

**Construction 5.1.** Construction of the perspective image of points on the observer's plane: Let  $P \neq O$  be a point on the observer's plane. Then the ray  $\overrightarrow{OP}$  crosses the equator at a point  $Q$ . Then the perspective image  $X$  of  $P$  will be at the equator of the perspective disc. Also, if  $\theta$  is the longitude of the meridian crossing  $Q$ , then  $\angle ENX = \theta$ . Hence the following construction plots the perspective image of  $P$ : make the perspective plane do double duty, using it to represent also the orthogonal back view of the observer's plane, with the image of  $N$  in the perspective view coinciding with  $O$  in the orthogonal view and the perspective scaled in such a way that the equator's perspective image coincides with its orthogonal image. Now plot  $P$  in the orthogonal image from its coordinates and trace the ray  $\overrightarrow{OP}$ . The point where  $\overrightarrow{OP}$  touches the equator will be the perspective image of  $P$ .

The problem of solving a scene can be divided into two parts: plotting points and lines in the anterior half-space and in the posterior half-space. Plotting inside the anterior half-space is solved in [1]. We will very quickly give a review of the method.

**5.1. Solving the anterior hemisphere.** It is shown in [1] that the perspective image of lines in the anterior half-space is well approximated by arcs of circle. This is important for two reasons: the first is that in drawing practice, arcs of circle are easy to trace with ruler and compass or even freehand; the second is that three points determine a unique arc of circle<sup>7</sup>.

On what follows we always assume that the lines being plotted do not cross  $O$  unless stated. (The case where it does is easy, the perspective image consisting of two antipodal points that we will also learn how to plot).

We have to consider two cases.

**5.1.1. Images of lines on a frontal plane.** We say that a plane is *frontal* if it is perpendicular to the central axis. Let  $l$  be a line on a frontal plane  $H$ . First suppose that  $H$  is not the observer's plane. Translating  $l$  to  $O$  we find it has two vanishing points which are diametrically opposite points on the equator. They are found by drawing the translated line directly on the perspective disc, to obtain its intersection with the equator (the perspective plane doing double duty as in construction 5.1). Next, we find a third point. If the line is not vertical, it intersects the sagittal plane at some point  $P$ . We measure the angle of  $\angle PON$  and plot the

<sup>7</sup>Through three different points  $P, Q, R$  on a plane there passes a single circle. To find that circle find the perpendicular bisectors of  $\overline{PQ}$  and  $\overline{QR}$  and intersect them at  $X$ , then open the compass from  $X$  to  $P$ .

measure of this angle on the vertical measuring line. If the line is vertical then it crosses the  $z = 0$  plane and we measure instead the angle with the central ray of sight at this point, and plot it on the horizontal measuring line. The image of the line is well approximated by the arc of circle that crosses the two vanishing points and the third point (see fig. 3).

Now suppose that  $H$  is the observer's plane. We get two vanishing points in the same way as above. The third point is obtained in the same way as before but will now be found on top of the equator of the perspective disc. The arc of circle will be one half of this disc.

Notice that measuring angles along the vertical plane and along the horizontal plane is very natural in practice, and the kind of thing you can do using an improvised theodolite when drawing from nature.

**Construction 5.2.** We note that we now know how to plot an arbitrary point located on the anterior half-space. Given a point  $P$ , consider the frontal plane going through  $P$  and on it a vertical line  $v$  and a horizontal line  $h$  going through  $P$ . We have just learned how to obtain the perspective images of these lines. The perspective image of  $P$  will be found at the intersection of the images of  $v$  and  $h$ .

**5.1.2. Images of receding lines.** We say that a line is a *receding line* if it intersects the observer's plane at a single point. Let  $P$  be the point of intersection of a receding line  $l$  with the observer's plane. We construct the image of  $P$  as in construction 5.1. Denote the image also by  $P$ . The plane  $H$  defined by  $O$  and  $l$  it must also intersect the equator at the antipodal point  $\bar{P}$ . To find a third point, we translate  $l$  to  $O$  and intersect it with the sphere to find the two vanishing points. One of these will be on the anterior hemisphere. We find it by construction 5.2 for plotting an arbitrary anterior point. Let its image be  $V$ . We trace the auxiliary arc of circle  $PV\bar{P}$  that is the image of the plane  $H$ . The image of  $l$  will be the part of the arc that lies between  $V$  and  $P$ . A case of particular interest is that of the central lines. We say that a line is central if it is perpendicular to the plane of the observer. In this case point  $V$  will project onto  $N$ , hence will be between  $P$  and  $\bar{P}$ . The image of  $H$  will be the straight line segment  $P\bar{P}$  and the image of  $l$  will be the segment  $\bar{P}N$  (see fig. 3).

This ends the review of (hemi)spherical perspective as presented in [1]. Outside of the anterior space disc the images of lines are no longer well approximated by circles, and that is perhaps why spherical perspective was kept limited to the anterior 180 degrees in its original formulation. However, as it turns out, the generalization is easily constructed both with ruler and compass and even in freehand drawing, by approximating the images of great circles on the posterior hemisphere by "fat lines" consisting of segments of circles, in a reasonably easy construction.

**5.2. The full 360°.** Let  $l$  be a spatial line. Together with  $O$ ,  $l$  defines a plane  $H$  through the origin, and this plane defines a great circle. We wish to plot, to a good approximation, the full perspective image of this great circle. The image of  $l$  will be contained in it and lie between its vanishing points.

What we want is to use the arc of circle approximation obtained in last section for the anterior region and use it to obtain a full plot of the great circle. The key lies in plotting antipodal points.

**5.2.1. Plotting antipodal points.**

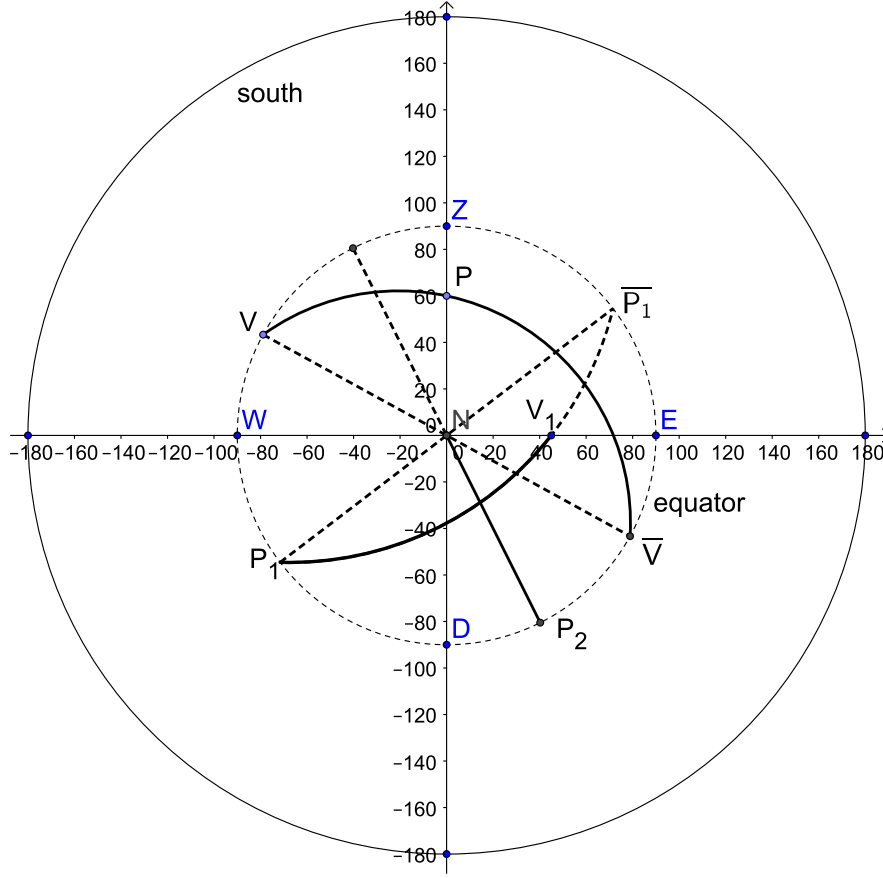


FIGURE 3. Perspective images of lines on the anterior hemisphere. Arc  $VP\bar{V}$  is the image of a frontal line. Arc  $P_1V_1\bar{P}_1$  is the image of the plane of a receding line, and the arc from  $P_1$  to  $V$  is the image of the line itself. The line segment  $\bar{P}_2N$  is the image of a central line. The image of its plane proceeds until the image of the antipode of  $P_2$ . See text for details.

**Proposition 5.3.** *Let  $P$  such that  $N \neq P \neq O$  be a point in space with antipode point  $\bar{P}$ . Let  $X$  and  $\bar{X}$  be the perspective images of  $P$  and  $\bar{P}$  respectively. Then  $\bar{X}$  is the point on  $\overrightarrow{XN}$  such that  $|X\bar{X}|$  equals the radius of the perspective disc.*

*Proof.* The plane  $H = NOP$  defines the single great circle  $C$  through  $N$  and  $P$ . It is the union of two diametrically opposite N-meridians,  $m_1, m_2$ . Assume  $P \in m_1$ . Then  $\bar{P} \in m_2$ . The images of  $m_1$  and  $m_2$  are diametrically opposite measuring lines  $r_1, r_2$ , respectively, that together form a diameter of the perspective disc. Let  $X, \bar{X}$  be the images of  $P, \bar{P}$  respectively. Then  $X$  is on  $r_1$ , and therefore  $\bar{X}$  is on  $\overrightarrow{XN} \supset r_2$ . Since  $\angle POP = 180^\circ$  is the length of a meridian and  $r_1, r_2$  are measuring lines, then  $|X\bar{X}|$  is the length of a measuring line, or a radius of the perspective disc.  $\square$

This proposition allows us to easily plot the antipode of an already plotted point  $X$ . Just draw a line to  $N$  from  $X$ , open the compass with radius equal to the radius of the perspective disc, and intersect with the line to find  $\bar{X}$ . Or, if using a marked ruler, pass the ruler through  $X$  and  $N$  with the zero marker at  $X$ , and plot the point at the mark of the length of the radius.

For the purposes of freehand drawing of a perspective it is often useful, when plotting points nearer to the equator than to  $N$ , to use instead the following result:

**Proposition 5.4.** *Let  $P$  such that  $N \neq P \neq O$  be a point in space with antipode point  $\bar{P}$ . Let  $X$  and  $\bar{X}$  be the perspective images of  $P$  and  $\bar{P}$  respectively. Let  $X_S$  be the intersection of  $\overrightarrow{XN}$  with the blow-up of  $S$ . Then  $\bar{X}$  is the point on  $\overrightarrow{XN}$  such that  $|\overline{XX_S}| = |\overline{NX}|$ . Also,  $|\overline{XN}| = |\overline{X(-X_S)}|$ , where  $-X_S$  is the point on the perspective disc diametrically opposite to  $X_S$ .*

*Proof.* The plane  $H = NOP$  defines a great circle  $C$  that contains  $P, \bar{P}, N$ , and  $S$ . On that plane, the lines  $\overline{P\bar{P}}$  and  $\overline{N\bar{S}}$  intersect at  $O$ , and therefore we have the equalities between opposing angles  $\angle PON = \angle \bar{P}OS$  and  $\angle POS = \angle \bar{P}ON$ . Since  $C$  is a great circle through  $N$ , it projects to a diameter of the perspective disc. On  $C$  we have a cyclic order of points  $P - N - \bar{P} - S$ . The perspective map is continuous outside of  $S$  and preserves the order on the image. We will have  $(-X_S) - X - N - \bar{X} - X_S$  where  $X_S$  and  $(-X_S)$  are the points of the blow-up corresponding to the directions of the two meridians of  $C$  at  $S$ . Because distances are preserved along measuring lines, the two angle equalities above imply  $|\overline{XN}| = |\overline{XX_S}|$  and  $|\overline{X(-X_S)}| = |\overline{XN}|$  respectively.  $\square$

The practical interest of this proposition lies in the fact that for freehand plotting of lines in the full spherical perspective it is often easier to transport the measurement  $|\overline{XN}|$  by eye than to transport the radius of the disc without an actual compass or ruler.

But, having a compass at hand, or a marked ruler, the use of proposition 5.2.1 makes for very efficient plotting of antipodes.

This allows us to plot the image of a great circle's posterior meridian from the image of its anterior meridian.

**Construction 5.5.** Construction of *fat lines* : Let  $C$  be the perspective image of the anterior meridian of a great circle on the sphere. To obtain an approximation of the posterior image of  $H$ , trace an arbitrary number of measuring lines  $m_1, \dots, m_K$  through  $N$ . Intersect each of these lines with  $C$  to get points  $Y_1, \dots, Y_K$ , and use proposition 5.2.1 to obtain the antipodes  $Y_i$ . Through each successive three of these points we trace an arc of circle, thus getting overlapping arcs  $Y_1Y_2Y_3, Y_2Y_3Y_4$ , etc. These overlapping arcs form a *fat line* that approximates  $H$ . The degree to which successive arcs fail to exactly overlap (how "fat" the envelope of these arcs is) indicates the amount of error in the approximation and the need to increase the number of measuring lines  $m_i$ <sup>8</sup> (see fig. 4).

**Remark 5.6.** The practical draughtsman, being given  $C$ , and armed with a marked ruler, will follow the following procedure: stick a nail on the center of the perspective disc, and sliding the ruler against the nail to ensure that it always touches  $N$ , make

<sup>8</sup>Alternatively just plot successive non-overlapping arcs and judge the error by how much the tangents differ at the endpoints of successive arcs.

its zero mark slide against along the curve  $C$ , plotting points at the mark of the length of the radius of the disc at desired intervals. With this procedure a great number of points can be marked very quickly, to the point where the antipodal curve can be interpolated by hand with good precision.

We are now ready to plot arbitrary lines in the full perspective. We have the following cases:

**5.2.2. Images of lines in frontal posterior planes.** Let  $l$  be a line in a frontal posterior plane. Let  $H$  be the plane defined by  $l$  and  $O$  and  $C$  its great circle. Suppose  $l$  is not vertical. Then it crosses the sagittal plane at a point  $P$ . The image of  $P$  will be  $X$  such that  $|NX| = \angle PON$ , on  $\overrightarrow{NZ}$  or on  $\overrightarrow{ND}$  according to whether  $P$  is above or below the observer. The antipode of  $P$  will map to the point  $\overline{X}$  on the same axis and at a distance equal to the radius of the perspective disc. This point will be in the inner disc corresponding to the anterior half space. Taking point  $\overline{X}$  and the two vanishing points at the equator we plot the approximation of the line through and arc of circle by the method of [1] described above. Let this arc be  $G$ . To obtain an approximation to the posterior image of  $C$  we can now trace an arbitrary number of measuring lines through  $N$  and follow the procedure of construction 5.5 to trace a fat line approximation of the posterior half of  $C$  from antipode points of  $G$ . Thus we obtain the full image of  $C$ . To obtain the image of the line  $l$  simply discard the anterior half of  $C$  (see fig. 5).

**Construction 5.7.** We can now locate an arbitrary point  $P$  on the posterior half-space: just pass vertical and horizontal lines through it, plot them according to the procedure just described, and intersect them to find the image of  $P$ .

**5.2.3. Images of receding lines.** Let  $l$  be a line that crosses the observer's plane at a single point  $P$ . Let  $H$  be the plane defined by  $l$  and  $O$  and  $C$  its great circle. Displacing  $l$  to the origin we obtain two vanishing points; one on the anterior hemisphere, Let it be  $F$ , the other on the posterior hemisphere, its antipode  $\overline{F}$ . Plot  $F$  according to the methods of [1], then use proposition 5.2.1 to plot  $\overline{F}$ . Let their images be called  $Y$  and  $\overline{Y}$ . Through construction 5.1 we obtain the points  $X$  and  $\overline{X}$ , the perspective images of  $P$  and its antipode respectively, which are on the equator. Trace the arc of circle  $XY\overline{X}$ . From that arc use construction 5.5 to trace the fat line of its antipodal arc. This plots the full image of the great circle  $C$ . To get the image of  $l$ , discard the arc of the curve defined by  $Y\overline{X}Y$  (see fig. 5).

In the particular case in which  $l$  is a central line, its great circle  $C$  is an N-meridian, hence its image will be a diameter of the disc. One of the vanishing points will be at  $N$  and the other will be split into two diametrically opposite points  $Y$  and  $-Y$  at the blow-up circle, each at the longitude of one of the two meridians from which  $S$  can be approached along  $C$ . Let  $X$  be the image at the equator of the point  $P$  where  $l$  crosses the observer's plane, which we can obtain by construction 5.1.  $X$  will either be on  $\overrightarrow{NY}$  or on  $\overrightarrow{N\overline{Y}}$ . Suppose it is the former case. Then the image of  $l$  will be the measuring line  $\overline{Y}N$ . Put in another way, the image of  $l$  will be the measuring line that contains the image of  $P$  (see fig. 5).

**5.3. plotting curves of constant angular elevation.** What we have just learned is enough to solve a scene when we have the cartesian coordinates of its points - for instance when drawing from an architectural plan. When drawing from observation, however, the artist is not in the position of the architect, but in that of the

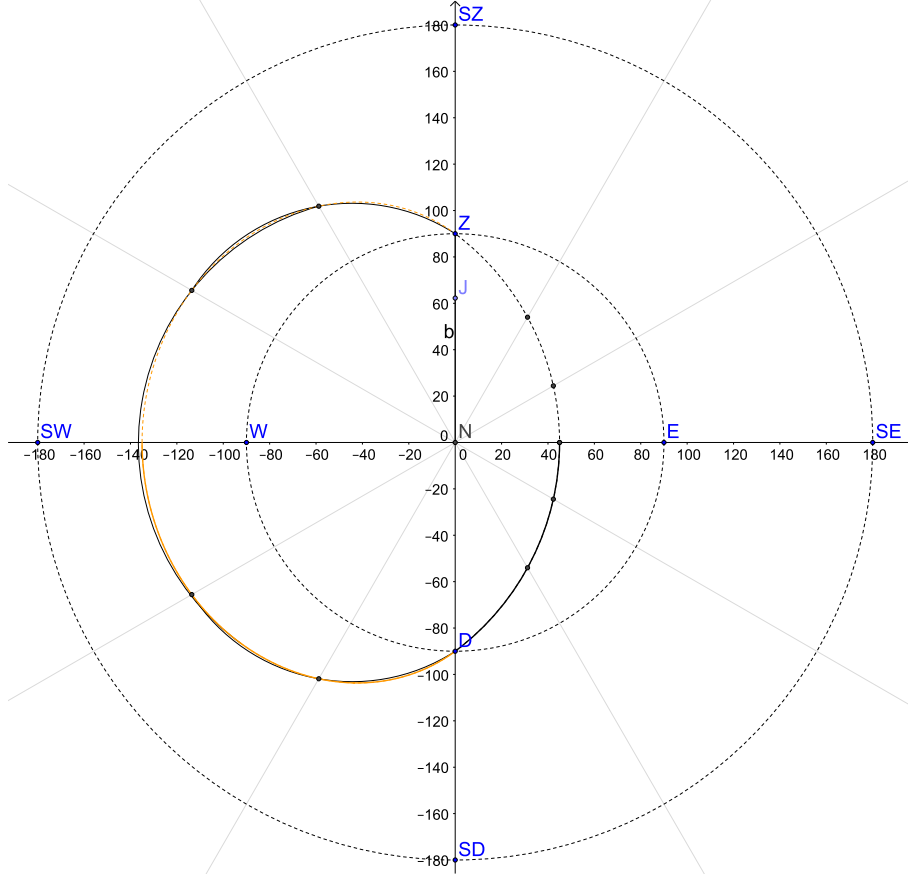
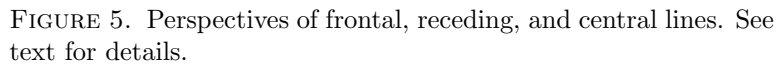


FIGURE 4. Perspective image of a vertical line placed at 45 degrees to the observer's right. The orange line is a point by point plot of the antipodes of the arc of circle on the anterior view. The black lines are a "fat line" approximation obtained from taking the antipodes from only two lines (and their mirror images), at 30 and 60 degrees with the horizontal axis. Even this coarse approximation fails at its worst point by little more than one degree.

astronomer. What he measures are the angles subtended by objects. Now we have already seen what the natural spherical coordinates are for this perspective (the angles  $\lambda$  and  $\theta$  defined above), and it is possible to construct a simple device to measure such angles directly, but the more habitual set of angles are the horizontal angle  $\xi$  together with the angular elevation  $\zeta$ , defined thus:  $\xi$  is the angle between the central ray  $\overrightarrow{ON}$  and the orthogonal projection of  $\overrightarrow{OP}$  against the horizontal plane.  $\zeta$  is the angle between  $\overrightarrow{OP}$  and its orthogonal projection on the horizontal plane. These are the angles one would measure with a standard theodolite.

Lines of constant horizontal angle  $\xi$  are the images of vertical lines and we already know how to plot them.



. Let  $h$  be parallel on the anamorphic sphere, of constant angular altitude  $\zeta$ .  $C$  intersects the equator in two points  $P_W$  and  $P_E$  on the West and East side of the sagital plane respectively and intersects the sagital plane in a point  $Q$ .  $P_W$  and  $P_E$  will be mapped to the equator at  $X_E$  and  $X_W$  respectively in such a way that  $\angle X_E O E = \angle X_W O W = \zeta$ .  $Q$  will be mapped to  $Y$  on the vertical line  $NZ$  in such a way that  $|NY| = \angle NOQ$ . We draw the arc of circle  $X_E Y X_W$  and take this as an approximation of the image of the parallel  $C$ .

<sup>9</sup>This is inconsistent notation, since in keeping with the geographical metaphor, parallels should be the circles of the planes perpendicular to the north-south axis, not the zenith-nadir axis. But it is a convenient term, and we will use it with apologies.

To plot the posterior part of the parallel we make use of the following proposition:

**Theorem 5.8.** *Let  $h$  be a parallel on the anamorphic sphere. Let  $P$  be a point of  $h$ , with perspective image  $X$ . Let  $U = \overrightarrow{NX} \cap \varepsilon$  where  $\varepsilon$  is the image of the equator. Let  $Y$  be the point such that  $U$  is the middle point of  $\overline{XY}$ . Then  $Y$  is the perspective image of a point of  $h$ .*

*Proof.* Parallels and N-meridians lines are both invariant with regard to reflection on the observer plane (because so are their defining planes and the sphere itself and hence their intersection). Then the intersection set of parallels and measuring lines is itself invariant for reflection on the observer plane. This intersection is made up of no points at all or of two symmetric points. Let  $h$  be a parallel and let  $P$  be a point on  $h$ . There is a single N-meridian  $m$  through  $P$ , and this meridian must cross the parallel at another point  $R$ . By mirror symmetry, if  $Q$  is the point where the meridian line crosses the equator, we must have  $\angle POQ = \angle QOR$ , and, since  $m$  is an N-meridian its image  $r$  is a measuring line through the origin and the points  $X, U, Y$  (image points of  $P, Q, R$  respectively) will verify  $|XU| = |YU|$ .  $\square$

. To plot the posterior half of a parallel  $h$ , plot first the anterior half  $h_a$  as an arc of circle, then plot a set of measuring lines  $r_i$ , intersect them with  $h_a$  at points  $X_i$ , find the points  $Y_i$  from proposition 5.8, and trace a fat line through the  $Y_i$ .

Figure 6.a) shows a computer plot of a uniform grid of parallels and verticals calculated directly from map 4.1. Figure 6.b) shows the approximation of the parallels of elevation 10, 45, 80, and 85 degrees plotted by the method above. It is quite evident that near the equator the approximation is not very good; the curves are not smooth at the transition from the anterior to the posterior hemisphere. This is an artefact of the approximations, as it is easy to see from equation 4.1 that the perspective images of constant elevation curves are differentiable. The error comes not from proposition 5.8, which is exact, but from the initial approximation of the parallel by an arc of circle inside the anterior disc. Near the equator the draughtsman might do well to avoid plotting vanishing points from parallels and verticals and use horizontals and verticals instead.

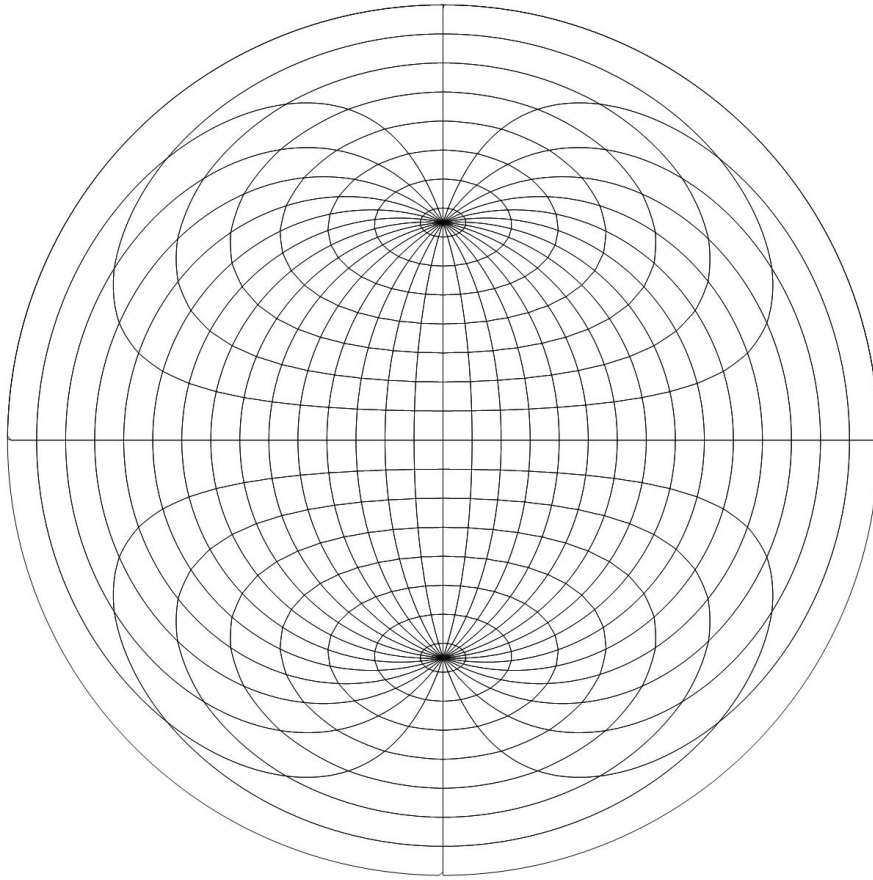
## 6. EXAMPLES

As is well known in classical perspective drawing, as long as we can plot a grid of squares we can plot any object to any required precision, by caging it inside a fine enough grid and interpolating through the intermediate points. We will therefore concern ourselves with the basic examples of grid construction.

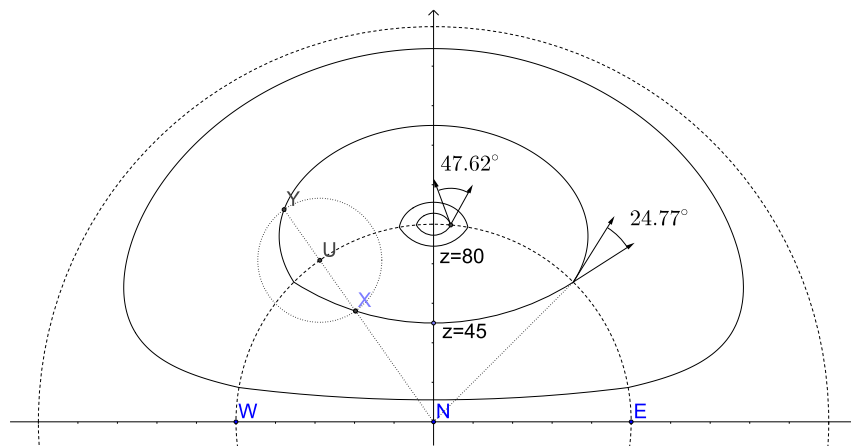
In fig. 7 we build the image of a central perspective grid<sup>10</sup>, i.e, we consider the perspective image of a horizontal grid of squares with one axis perpendicular to the observer's plane. We assume for simplicity that a vertex is directly under the observer, hence one axis is directly under the  $EW$  axis of the sphere. The plane of perspective does triple duty here, serving also to represent a top orthogonal view of the scene and a back view of the observer's plane. We take the sphere's arbitrary radius to be equal to the height of the observer relative to the ground plane, so that a horizontal line through the nadir point  $D$  coincides with an axis of the grid and will be made to represent both the ground plane on the back view

<sup>10</sup>This is what in classical perspective would be called a 1-point perspective grid but that would be a misnomer in the spherical case since we get to plot the vanishing points of the frontal lines.





(A) Grid of verticals and lines of constant elevation, plotted directly from function 4.1.



(B) Ruler and compass approximation of curves of constant angular elevation of 10,45,80, and 85 degrees. Note the break of differentiability at the equator.

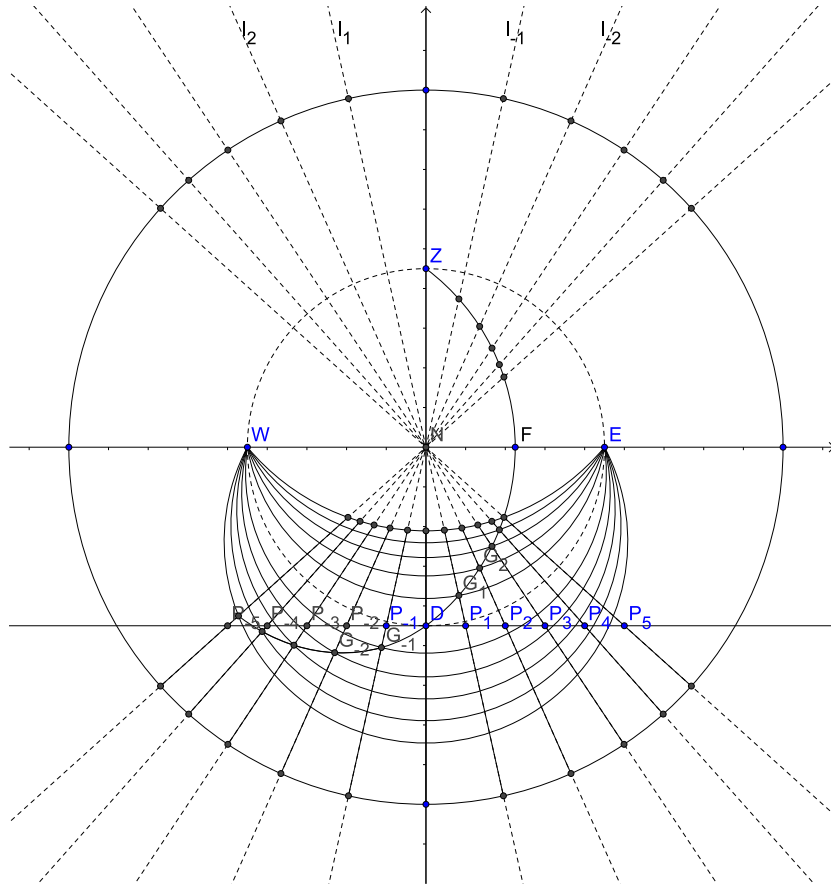
FIGURE 6. Lines of constant angular elevation

and the observer's plane on top view. On the top view the receding lines of the grid intersect the ground line of the observer's plane at uniformly spaced points  $P_i$ . The perspective of each such point is obtained by intersecting the line that joins them to  $N$  with the equator. The same straight line is the perspective image of the central receding line of the grid that crosses that point. Thus the image of the receding lines of the grid is a set of straight lines  $l_i$  going from  $N$  to the blow up and passing through the  $P_i$ . Note that this is exactly the same construction as in classical perspective, though with a different interpretation.

To plot the frontal lines we first trace a line  $r$  on the plane of the grid, such that it makes a 45 degree angle to the right of the observer and crosses the nadir point. This line will cross each row of squares, touching a vertex of each row. We plot the great circle  $C$  of the plane defined by  $O$  and  $r$ . First we plot the anterior part by drawing the arc  $DFZ$  where  $F$  is the 45 degree vanishing point on the horizontal axis. Then we note that each of the receding lines  $l_i$  crosses a vertex of the grid where it touches the anterior part of the 45-degree line. To find these points we intersect each  $l_i$  with the upper half of  $C$  on the anterior disc and take the antipodal point of that intersection. Since this point will be on the posterior space and will belong to  $r$  and to  $l_i$ , then it will be the required vertex. So we can plot a posterior horizontal line through it to get a posterior planar line of the grid. Note that we have used the receding lines of the grid as the natural measuring lines to construct the posterior fat line of  $r$ . In this fashion we can plot the full 360 degree grid to any required precision and extension.

In fig. 8 we represent a tiled cubic room drawn from the point of view of an observer at its center, looking straight into the center of one of the walls. The whole setup is drawn very simply from a judicious use of vertical and horizontal frontal lines at 45 degrees to the observer; these lines do double duty, as, for instance, the frontal vertical at 45 degrees to the right of the observer has the same projection as the line on the plane of the horizon that goes from the nadir to the 45 degree mark on the  $EW$  measuring line.

Often we will want our grids to be oriented at some arbitrary angle to the ray of sight. In fig. 9 we represent a square  $ABCD$  on a horizontal plane, below, behind, and to the left of the observer, such that one side of the square makes a 60 degree angle with the ray of sight. Once again the plane of perspective does triple duty, serving also to represent a top orthogonal view of the scene and a back view of the observer's plane and we take the sphere's arbitrary radius to be equal to the height of the observer relative to the ground plane, so that a horizontal line through the nadir point represents both the ground plane on the back view and the observer's plane on top view. On this top view we draw the square  $ABCD$  and project its sides until they intersect the line of the observer's plane. From the back view we can now trace lines to these intersection points and find their projections on the equator. From these projections and the vanishing points we can find the arcs of circle representing the lines that extend the sides of the square (note that the vanishing points are all on the horizontal measuring line, one set of lines converging to the points at  $60^\circ$  and  $-120^\circ$  and the other to  $-30^\circ$  and  $150^\circ$ ). From the arc of the anterior perspective we can obtain the corresponding fat lines of the posterior perspective. By intersecting these lines we find the perspective images of the points  $A, B, C, D$ . From this square we can plot a grid by an adaptation of the previous methods.



## 7. COMPARISON WITH REFLECTIONS ON A SPHERE

It is natural to ask if it is indeed a reflection.

We notice several difficulties. Given  $R$ , it is easy to find the incident and reflected rays, but the inverse problem of obtaining  $R$  from  $P$  is non-trivial. In general it requires solving an algebraic equation of order four (see [2]).

<sup>11</sup>An enterprising artist, more concerned with speed of execution than exactness might use it for an easy substitute of a true sphere reflection; the casual viewer might very well not notice the difference.

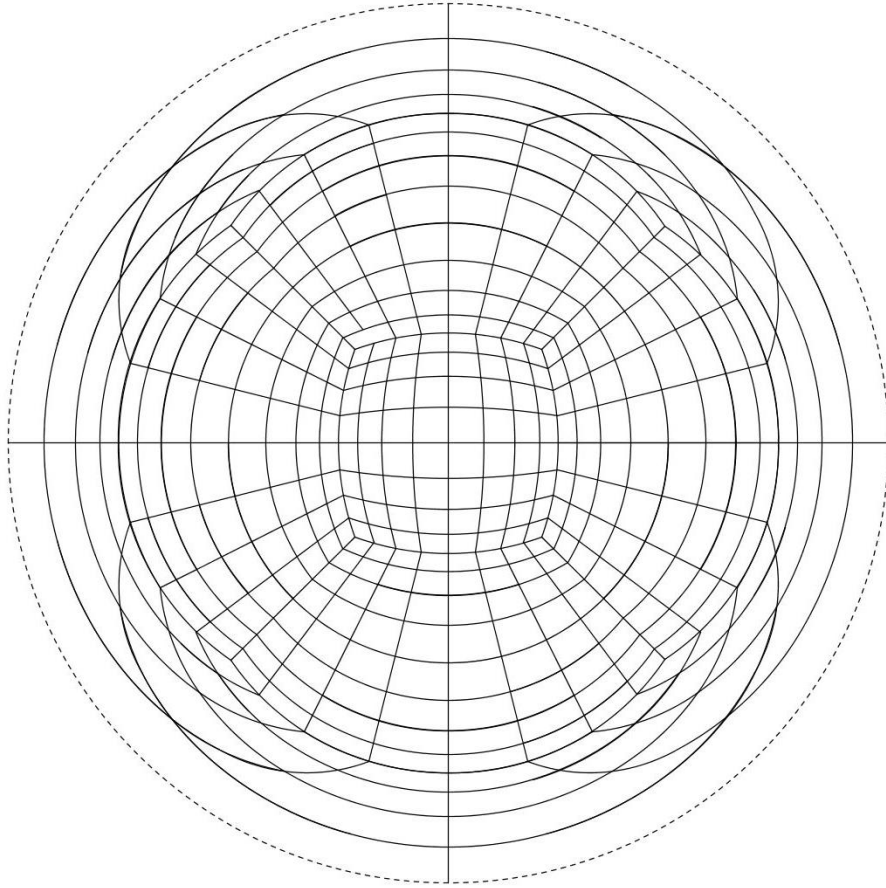


FIGURE 8. Room forty-five.

either the center  $O$  or the observer  $E$ . By contrast, obstructions are always radial for perspectives, being fully determined before the flattening, at the step of the anamorphosis.

Finally, our spherical perspective has an angle of view of  $360^\circ$ . The angle of view captured by a reflection depends on the distance of the observer to the sphere. The points of the sphere define a cone with the observer  $E$  at the vertex, the cone of shadow, and every point outside of this cone of shadow will be viewable on the sphere. In angular terms, the field of view will be  $360^\circ - \delta$  with  $\delta = -2 \arctan(r/d)$ , where  $r$  is the radius of the sphere and  $d$  the distance of the observer from the center of the sphere.

There is however a limiting case wherein these problems disappear.

There are two ways of making  $\delta$  go to zero: you can move away from the sphere (and look at it from a telescope to compensate) or stay put and shrink it (and look at it with a microscope), or some combination of the two that will make the  $r$  negligible small compared to  $d$ . Either way will, in the limit, make for a  $360^\circ$  angle of view (the first option will leave a tubular shadow of finite section, however, while the second will reduce the shadow to a ray).

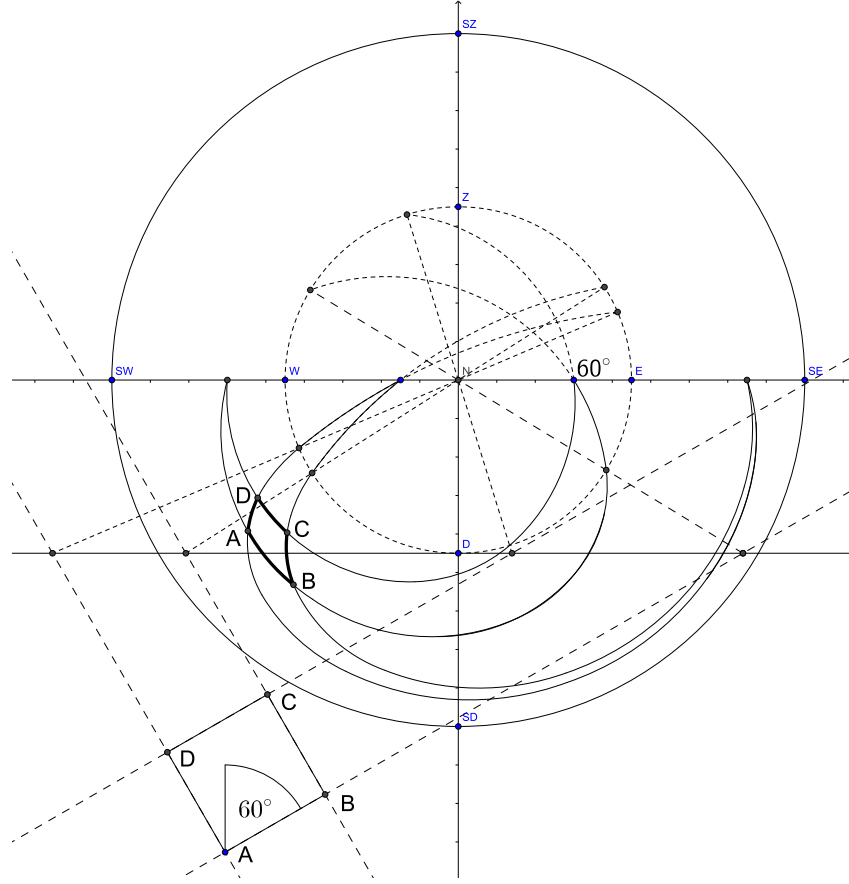


FIGURE 9. A square, below, behind, and to the left of the observer.

In any case, in the limit  $r/d \rightarrow 0$ , the rays coming from the observer  $E$  to the sphere with center  $O$  become parallel to  $\overline{EO}$ , and the angle of reflexion  $\alpha$  becomes equal to the angle  $\beta = \angle EOR$ . Hence  $\angle ERP = 2\alpha$  (see fig. 11).

Still, the position of  $P$  is not determined by  $\beta$  but by the angle  $\lambda$  of fig. 10. We can however make these angles equal by either restricting the reflection to objects at infinite distance (say, plotting a reflection of the celestial sphere) or by making  $r$  go to zero. When  $r$  goes to zero,  $R$  goes to  $O$  and  $\lambda$  becomes  $2\alpha$ . In that limit, the projection becomes radial (therefore making obstructions trivial), and the whole space of directions is mapped onto the hemisphere visible from  $E$ . This can be seen as a sphere anamorphosis followed by a linear contraction onto a hemisphere by halving the angle  $\lambda$  of each point on the sphere.

Seen from point  $E$ , since all rays are parallel to the axis  $OE$ , the reflection will look like the orthogonal projection along  $OE$  of the image on the sphere. Hence the reflection, seen from  $E$ , is anamorphically equivalent to a perspective obtained by the trivial anamorphism onto the sphere composed with a flattening which is the composition of a linear compression onto a hemisphere followed by an orthogonal

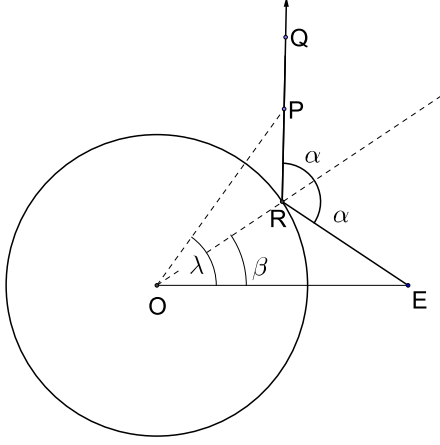


FIGURE 10. Non-radial obstructions. Points P and Q both project to R although they are not in the same ray from O.

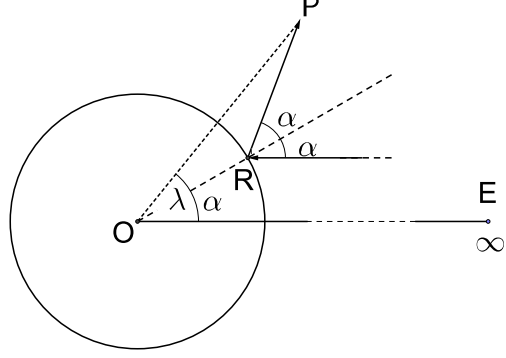


FIGURE 11. With E at infinity all rays become parallel and angle  $\beta$  becomes equal to  $\alpha$ . If P goes to infinity then  $\lambda$  goes to  $\alpha$ .

projection. In the spherical coordinates of 4.2 (with the  $y$  axis on  $\overrightarrow{OE}$  and  $x, z$  in the perpendicular plane through  $O$ ) and rescaling the sphere to  $r = 1$ , this perspective is the map

$$(\rho, \lambda, \theta) \mapsto (1, \lambda, \theta) \mapsto (1, \lambda/2, \theta) \mapsto \sin(\lambda/2)(\cos(\theta), \sin(\theta))$$

where the first map is the trivial anamorphosis, the second is the crunching into the hemisphere and the last step is the orthogonal projection.

This is a 360 degree perspective, but different from our spherical perspective. It is not linear along  $\lambda$ , squashing the outer angles more, and cannot be easily used for drawing by hand without the help of pre-computed grids (since we lose the isometry along measuring lines). But we can see why there is a qualitative similarity between the two.

It has been noted in [2] that reflections on a sphere could be used as a form of wide angle "perspective". This is well inspired in art history, in fact, as reflections drawn from observation have been the time honoured tool of the artist to represent a wide angle of view, Escher's self portrait being a well known example.

We have already seen the difficulties in this approach. First, reflections are hard to calculate. Second, they are not perspectives in our sense of the word (i.e., "radial" perspectives), and they have non-trivial obstructions. As noted in [2] this causes difficulties for hidden-face removal algorithms.

If the purpose is to represent a wide angle view, spherical perspective is a much more natural proposal. It allows for (up to) a 360 degree view; it is a perspective in the sense we defined above and therefore, like all such perspectives, has trivial obstructions, with hidden-face algorithms working exactly as in the classical case, being calculated at the anamorphosis step. Furthermore, spherical perspective is

easy to calculate by map 4.1 and, unlike the limiting case of reflections mentioned above, it can actually be used by an artist armed only with ruler and compass or even, after some practice, in freehand drawing from nature.

Note: Further notes, computer code and illustrations will be made available at the author's page:

<http://www.univ-ab.pt/~aaraujo/full360.html>

## 8. ACKNOWLEDGEMENTS

This work was supported by Fundação para a Ciência e a Tecnologia (FCT) project UID/MAT/04561/2013.

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